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MODELS FOR PACKING CIRCULAR OBJECTS INTO RECTANGULAR SPACES

Serano L. Oryan

College of Arts and Sciences,
Benguet State University

ABSTRACT

Novel mathematical models for packing identical circular objects into rectangular spaces are here presented. The study explores on different packing patterns that tend to increase the population density of a given rectangular space by way of systematic repositioning of the objects and by applications of some trigonometric concepts in determining the effect of repositioning to the vertical distances between the centers of the objects across the contiguous rows. The results showed that if the dimension (rc) of a rectangular space is rc = 8x5, where the unit of measure of the space is the diameter of a circular object, then the default arrangement of the objects can be repositioned so that the content of the space is maximum. The results also showed that a rectangular space attains its maximum content if row, r, is a multiple of $[7.464]$ and column $c \geq 5$. In order to determine whether the population density of a rectangular space can be increased by applying some packing patterns, two mathematical models are developed, through which the exact number of objects that can be accommodated in a space is calculated. This study shows that there are deterministic mathematical models of calculating the maximal number of identical circular objects that can be packed into rectangular spaces. In cases, however, where the rectangular container provides empty space either on the row or column or both with length less than the diameter of one circular object, then adjustment on the models may be made. Hence, it is recommended that such particular cases have to be further explored in future study.

Keywords: *packing models, circular objects and rectangular spaces*

INTRODUCTION

As the 21st century unfolds, we see the rapid evolution of human civilization whose landscape is primarily defined by revolutionary breakthroughs in science and technological development *vis-à-vis* population explosion. As human population increases day by day resulting to rapid increase in the demand for the use of machines, appliances and gadgets of all kinds that tend to occupy significant livable space at home and to the rapid depletion of available arable space, technological advances have been on the development of machines and gadgets that address space saving concerns and multiple capability needs. Consequently, many machines, appliances and gadgets are now built with minimum sizes and with multiple task capabilities. Television sets, computers, communications and musical gadgets are few examples of technological products that have been miniaturized to save on space but with built-in capabilities far exceeding their much bigger sized predecessors.

The idea of saving space in technological development has been extended to many human activities such as maximizing land use, increasing population densities of objects, maximizing content of boxes, *etc*.

This study attempted to address a specific concern on maximizing content of rectangular spaces when populated with circular objects.

It is said that one of the most scientifically challenging problems in operation research is packing circular and spherical objects in some predetermined rectangular two and three dimensional spaces. This is partly because circular objects do not tessellate unlike other objects with straight edges. Packing circular and spherical objects are real world activities being undertaken in industries involved in the production, packing and transport of such products as textile, sports equipment, automobile parts, food and hardware products, to mention a few. Application of packing patterns may also be extended to agriculture. The population density of a rectangular planting area may be increased by applying certain packing pattern without compromising the required circular area allotted to each plant.

Packing circular objects is considered a "very interesting NP-hard combinatorial optimization problem" because there is "no procedure [that] is able to exactly solve them in deterministic polynomial time" (Hifi and M'Hallah, 2009).

There were several researches conducted on packing circular objects. In a literature review on circle and sphere packing problems, Hifi and M'Hallah (2009) identified and described the most recent studies on the topic. These include the works of the following researchers: Stoyan (2003) on "Mathematical methods for geometric design", Stoyan (2004) on "A mathematical model and a solution method for the problem of placing various-sized circles into a strip", Szabo *et al.* (2007) on "New Approaches to Circle Packing in Square", Hifi and M'Hallah (2007 and 2009) on "A dynamic adaptive local search algorithm for the circular packing problem" and on "Beam search and non-linear programming tools for the circular packing problems". Locatelli and Raber (2002) stated "Packing equal circles in a square", Maranas *et al.* (1995) on "New results in the packing of equal circles in a square", Boll *et al.* (2000) on "Improving dense packings of equal disks in a square", Markot and Csendes (2005 and 2006) on "A new verified optimization technique for the packing [of] circles in a unit square problems" and on "A reliable area reduction technique for circle packing problems", Correia *et al.* (2000 and 2001) on "A new upper bound for the cylinder packing problems" and "Cylinder packing by simulated annealing", Birgin *et al.* (2005) on "Optimizing the packing of cylinders into a rectangular container: a nonlinear approach", George *et al.* (1995) on "Packing different-sized circles into a rectangular container", Hifi *et al.* (2004) on "A simulated annealing approach for the circular cutting problem", Lubachevsky and Graham (2009) on "Minimum perimeter rectangles that enclose congruent non-overlapping circles" and, among others, Huang *et al.* (2005) on "Greedy algorithms for packing unequal circles into a rectangular container".

The works of the above-mentioned researchers explored on possible ways of packing circular objects by determining the radii of circles that either optimize or maximize a given rectangular as well as circular container. Some of these works include determining the minimum perimeters of rectangles that enclose identical non-overlapping circles. The researchers developed several physical as well as electronic approaches of solving various packing circles problems ranging from "computer-aided optimality proofs, to branch-and-bound procedures, to constructive approaches, to multi-start nonconvex minimization, to billiard simulation, to multiphase heuristics and metaheuristics" (Hifi and M'Hallah, 2009). These approaches were applied to packing problems involving both identical and varied-sized circular objects.

Although the published paper presents thorough descriptions of the reviewed approaches, there was no illustration made showing how the approaches could be used to solve some particular packing problems.

This work presents a simple investigation on a special circle packing problem that deals with identical circular objects with fixed sizes and rectangular spaces whose dimensions' unit of measure is the diameter of a circle. This special packing problem was not thoroughly considered in the above studies. In the foregoing studies, various sophisticated approaches were developed in solving circle packing problems by focusing on minimizing the length of the radii of both identical and non-identical circles in order to attain optimum covers of the interiors of given rectangular spaces. Although there are studies that deal with packing identical non-overlapping circles in squares, the aspect considered is the circles which were randomly scattered into the square container. This study on the other hand, explored on non-random and smooth patterns of fitting the most number of identical fixed-sized circles in some pre-determined dimensions of rectangular spaces.

To ventilate the idea pursued in this study, consider the given rectangular packs above that are fully filled with circular objects in their default positions. The letters R and C denote the row and column of the rectangular space, respectively.

The dimension of the rectangular space in figure 1(a) is rc = 8x4, for figure 1(b), 5x9 and for figure 1(c), 8x6.

If the objects are rearranged in ways different from their original positions such that no two contiguous rows are retained in their original positions, is it possible for these packs to exceed their original contents? The primary aim of this study is to develop some mathematical models that can be used to calculate the optimal number of equal-sized circular objects that can be fitted in a given rectangular spaces.

METHODOLOGY

The nature of the study is exploratory. The default positions of the circular objects in a two dimensional rectangular space (or rc space) require that the objects are arranged in rectangular position in which the objects are placed along the straight lines in both row r and column c. The default position is altered, or repositioned, in some ways to form some new packs. The default pack is denoted by P_o while a new pack by P_n.

The structural positions of the circles in P_n are mathematically analyzed in terms of determining the effect of the alteration on the vertical distance between the centers of the circles across any two contiguous rows in r. The effect of the alteration on the content of P_n, in consideration of the dimension of rc space, is also explored.

In order to determine whether P_n is tending to maximize the number of circles in rc space or tending to minimize it, the content of P_n is compared with that of P_o. The minimum dimension of rc space that allows the addition of a layer in row r and that makes P_n exceeds the content of P_o is explored. Finally, the mathematical models of computing the contents of P_n are developed and proofs for their mathematical viability are provided.

RESULTS AND DISCUSSION

Given a fixed dimension of an rc space, the default pack (Fig 2a) when altered in certain way may yield the new pack shown in figure 2(b). The repositioning of the circles in figure 2(b) is a kind that tends to increase the number of layers in row r and the density at the middle part of the pack. This form of repositioning requires arranging the circles in triangular formation.

The effect of repositioning the circles in the manner as in figure 2(b) is row compression. In the default position shown in figure 3(a), the distance between the centers of the circles across any two

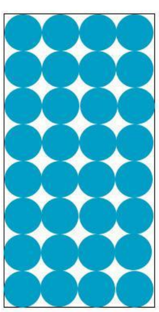
R

C

Figure 1(a)

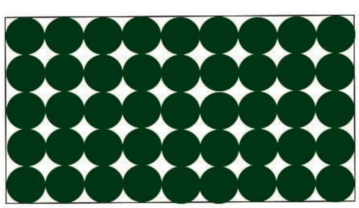
R

C

Figure 1(b)

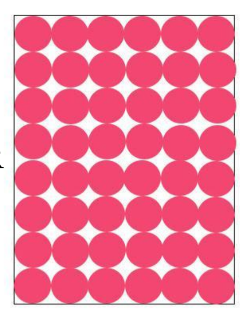
R

C

Figure 1(c)

contiguous rows is equal to the diameter of a circle. In the new position wherein each circle lies in between two circles below it, the vertical distance of the centers in the default position becomes a slant distance in the new position, as shown in figure 3(b) above. By applying tangent function in trigonometry it can be shown that the vertical distance (d_n) of the centers of the circles in their new position is **√3 /2 d**, which is 13.4% shorter than the original vertical distance (d_o) of the centers across rows. This result means that, except in the first row, about 13.4% of the original distance is lost in every row due to repositioning. Also, the result indicates that there exist a number *n* of rows in P_o in which the accumulated loss of distance will create enough space that will allow the addition of one more row in P_n.

In order to determine this, the difference between the height (H_o) of P_o and the height (H_n) of P_n, for some numbers *n* of rows, must be equal to the height (d_n) of the row to be inserted, which is √3 /2 d.

The height (H_o) of P_o is the product of the number of rows (n) and the diameter (d) of a circle. Thus,

$$H_o = nd_o = nd \quad (1)$$

Similarly, the height (H_n) of Pn is the total number (n-1) of shortened vertical distance (d_n) times length of dn which is √3/2 d plus the sum of the two un-shortened vertical distances which are the bottom radius (1/2d) of the first row and the top radius (1/2d) of the last row of circles in P_n. Thus,

$$Hn = (n-1)(\sqrt{3}/2d)+(1/2d+1/2d)=(n-1)(\sqrt{3}/2)+d \quad (2)$$

Finally, the number of rows in P_o that allows the addition of one more row in P_n is given in theorem 1.

Theorem 1: Let n∈Z be the number of rows in P_o and T∈Z be the minimum number of rows in P_o that accumulates a height enough to accommodate one more row in P_n. Then

$$T = \lceil n \rceil = \lceil 7.464 \rceil = 8$$

Proof:

When the difference between H_o and H_n is equal to the height (√3 /2 d) of the row to be inserted, then a row can be added to P_n. Thus,

$$H_o - H_n = \sqrt{3}/2d$$

Where: $\sqrt{3}/2\,d$ is, as shown above, the height (d_n) of a row in P_n.

$$nd - [n-1)(\sqrt{3}/2d)+d] = \sqrt{3}/2d \text{ From (1) and (2)}$$
$$n = 2(2-\sqrt{3})$$
$$= 7.464 \quad \text{(on solving)}$$

So, the number of rows in P_o that accumulates a

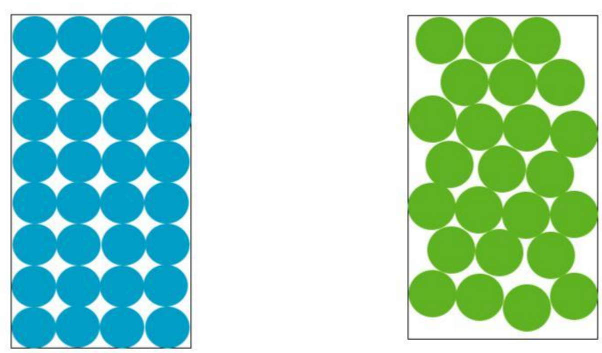


Figure 2(a)Figure 2(b)
Default Pack P_o New Pack P_n

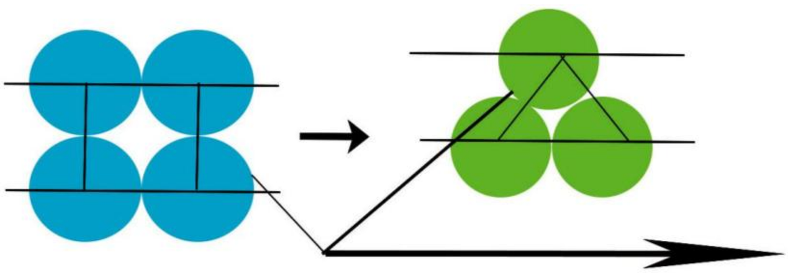


Figure 3(a) Figure 3(b)

height of √3/2d in P_n is 7.464. This is the minimum number (T) of rows that will allow the insertion of one more row in P_n. However, since T is an integer, then there exist $n_1, n_2 \in Z$ (where $n_2 = n_1+1$) such that if $n_1 < n < n_2$, then either $T = n_1$ or $T = n_2$. Because $n_1 < n$ implies that $T \neq n_1$. So, $T = n_2$. T is the least integer ≥ n. Thus,

$$T = \lceil n \rceil = \lceil 7.464 \rceil = 8$$

From the result above it can be concluded that the number of rows that can be repositioned to create enough space for the addition of a desired number, x, of rows is the least integer greater than the multiple of 7.464. This assertion is given in theorem 2.

Theorem 2. If x is the desired number of rows to be added to P_n and T_x is the total number of rows in P_o that can be repositioned to make the addition of x rows possible, then,

$$T_x = \lceil x(7.464) \rceil \text{ where } x = 1,2,3\ldots$$

Proof:

Since the addition of a row is possible only every time 7.464 rows are reached then the addition of x rows is possible only when 7.464 rows are reached x times. Thus the ratio and proportion below hold for any x:

$$\frac{1}{7.464} = \frac{x}{T_x} \Rightarrow T_x = x(7.646)$$

$$= \lceil x(7.646) \rceil, \text{ since } Tx, x\ Z+$$

As illustrations, if one wishes to know how many rows are there in P_o that can allow the insertion of 1, 2 and 3 more rows in P_n, then the desired totals are given below:

For x =1: $T1 = \lceil x(7.646) \rceil$ 14 15

For x = 2: $T2 = \lceil 2(7.464) \rceil = \lceil 14 .928 \rceil 23 = 15$

For x = 3: $T2 = \lceil 3(7.464) \rceil = \lceil 2 .393 \rceil = 23$

These illustrations show that the number of rows that can be inserted into a pack containing from 8 to 14 rows is the same and likewise the number of rows to be inserted into a pack containing from 15 to 22.

In general, the number of rows to be inserted in P_n is the same for the interval, $[\lceil \mathbf{x}(7.464)\rceil] \leq r < [\lceil (x+1)(7.464)x \rceil]$, where r is any number of rows in the interval.

Number of Circles that can be Contained in P_n after Repositioning

The result above can be extended so that whenever the dimension of P_o is known, then the maximum number of circles that can be fitted to P_n can be determined.

The formula of computing the maximum number of circles to be fitted in P_n, based on the dimensions of P_o, is given in theorem 3.

Theorem 3. Let r and c be the number of rows and columns in P_o; x, the number of rows that can be inserted in P_n; *m*, the number of rows with maximum number of circles in r; and *k*, the number of rows with maximum number of circles in x. Then the total number (T_n) of circles that can be fitted in P_n is

$$Tn = \begin{cases} \frac{1}{2}(r + x)(2c-1), & \text{if r and x are of the same parity} \\ \frac{1}{2}[(r + x)(2c - 1) + 1], & \text{if r and x are of different parity} \end{cases}$$

$$\text{Where } x = \frac{r}{7.464}$$

Proof:

Case 1. When r and x are of the same parity

If r and x are odd, the number (m) of rows with maximum number of circles in r is greater by one than the number (r-m) of rows with fewer circles; while in x, the number (k) of rows with maximum number of circles is one less than the number (x-k) of rows with fewer circles. Thus,

$m = (r-m)+1 = \frac{1}{2}(r + 1)$ and $k=(x-k)-1=\frac{1}{2}(x - 1)$

In r, the number of circles in an odd row is mc and in an even row is (r-m)(c-1). Similarly, in x, the number of circles in an odd row is kc and in an even row is (x-k)(c-1). Let Tn be the total number of circles in P_n. Then,

$$T_n = [mc + (r-m)(c-1)] + [kc)+(x-k)(c-1)]$$
$$= (mc + rc - r - mc + m)+(kc + xc - x - kc + k)$$
$$= [rc - r+(r+1)/2]+[xc - x+(x-)/2], \text{ since}$$
$$m = \frac{1}{2}(r + 1) \text{ and } k = \frac{1}{2}(x - 1)$$

$\mathbf{T_n = \frac{1}{2}(r + x)(2c-1)}$ on solving

Where $x = \frac{r}{7.464}$

Similarly, if r and x are both even numbers then the number of odd rows is equal to the number of even rows in both r and x. Thus,

$m = (r-m) = r/2$ and $k = (x-k) = x/2$

Consequently, the value of T_n is derived similarly as above.

$$T_n = [mc + (r-m)(c-1]) + [kc) + (x-k)(c-1)]$$
$$=(mc+rc-r-mc+m) + (kc + xc - x - kc + k)$$
$$= (rc - r + m) + (xc - x + k)$$
$$= (rc - r + r/2) + (xc - x + x/2),$$
since $m = r/2$ and $k = x/2$

$\mathbf{T_n = \frac{1}{2}(r + x)(2c-1)}$ on solving

Case 2. When r and x are of different parity

If r is odd and x is even then the number of rows which contain the maximum number of circles is greater by one than the number of rows with fewer circles in r. Also, the number of rows in x that contain the maximum number of circles is equal to the number of rows with fewer circles. Thus,

$m = (r-m) + 1 = (r + 1)/2$ and $k = (x - k) = x/2$ and the total number T_n of circles in P_n is

$$T_n = [mc + (r-m)(c-1)] + [kc + (x-k)(c-1)]$$
$$=(mc+rc-r-mc+m)+(kc+xc - x - kc + k)$$
$$= [rc - r + (r + 1)/2] + [xc - x + x/2)$$

$\mathbf{T_n = \frac{1}{2}[(r + x)(2c - 1) + 1]}$ on solving

If r is even and x is odd, then the number of even rows is equal to the number of odd rows in r; while in x the number of even rows is less by one than the number of odd rows. Thus,

$m=(r-m)=r/2$ and $k=(x-k)+1=(x+1)/2$ and

$$Tn = [mc + (r-m)(c-1)] + [kc + (x-k)(c-1)]$$
$$=(mc+rc-r-mc+m)+(kc + xc - x - kc + k)$$
$$= (rc - r + m) + (xc - x + k)$$

$\mathbf{Tn = \frac{1}{2}[(r + x)(2c - 1) + 1]}$ on solving

Maximum Number of Extra Circles That Can Be Inserted In P_n

The maximum number, T_c, of extra circles that can be inserted in P_n, based on the knowledge of the content of P_o, can be determined. The procedure of computing the value of T_c is given in Theorem 4.

Theorem 4: If r and c are the rows and columns of P_o, T_o is the number of circles in P_o and x is the number of rows that can be added to P_n, then the total number (T) of circles that can be inserted to P_n, given that $T_o{}^c = rc$, is

$$T_c = \begin{cases} \frac{1}{2}[x(2c-1) - r], & \text{if r and x are the same parity} \\ \frac{1}{2}[x(2c-1)+(1-r)], & \text{if r and x are of different parity} \end{cases}$$

Where $x = \frac{r}{7.464}$, as before.

Proof:

The total number, T_c, of circles that can be inserted into P_n is the difference between the contents of P_o and P_n.

Case 1: When r and x are of the same parity

Since r and x are of the same parity, the value of T_n, according to Theorem 3, is $\frac{1}{2}(r + x)(2c - 1)$.

Consequently, the value of T_c is

$$T_c = T_n - T_o$$
$$= \frac{1}{2}(r + x)(2c - 1) - rc$$

$\mathbf{T_c = \frac{1}{2}[x(2c - 1) - r]}$ on solving

Case 2: When r and x are of different parity

Since r and x are not both odd integers, T_n is equal to $\frac{1}{2}[(r + x)(2c-1)+1]$, according to Theorem 3.

Thus,

$$T_c = T_n - T_o$$
$$= \frac{1}{2}[(r + x)(2c - 1) + 1] - rc$$

$\mathbf{T_c = \frac{1}{2}[x(2c - 1) + (1 - r)]}$ on solving

Corollary 1: A positive value of T_c gives the number of extra circles that can be inserted into P_n.

Corrolary 2: A negative value of T_c gives the number of circles in P_o that cannot be fitted in P_n.

Corollary 3: A zero value of T_c means that the number of circles present in both P_o and P_n is the same.

Minimum Number of Circles in a Row That Allows the Insertion of an Extra circle

The addition of some extra circles is possible only if the number of rows in P_n exceeds that of P_o. However, if the number of circles in the extra rows added is less than or equal to the number of circles displaced by the repositioning, then addition of some extra circles does not happen. Therefore it is important to determine the minimum number of circles in a row of P_o that allows the addition of an extra circle whenever an extra row is added to P_n.

Theorem 5. If r is the minimum number of rows in P_o such that

$$\frac{r}{7.464} = 1$$

and c is the minimum number of circles in the row of P_o that allows the addition of one more circles to P_n, then c = 5.

Proof:

By hypothesis, the minimum number (r) of rows that satisfies the given equation is 8, a number that is shown previously to be also the minimum number of rows that allows the addition of one more row (x = 1) in P_n. Theorem 4 is used to compute the minimum number of circles that allows the addition of one more circles in P_n by letting T_c equal to 1. Thus,

$T_c = ½ [x (2c – 1) + (1- r)]$, since r and x are of different parity.

$1 = ½ [1 (2c – 1) + (1 – 8)]$, since x = 1, r = 8, and Tc = 1

5 = c, on solving

This means that when P_o contains 8 rows and 5 columns one more circle can be added to P_n.

Maximum Number of Circles That Can Be Fitted to a New Pack (P_m) by Partial Repositioning

In the previous manner of repositioning the circles, addition of a row in P_n is possible every time 7.464 rows are reached in P_o, as shown in theorems 1 and 2, and for these given rows addition of a circle in P_n is possible every time a column is added in excess of four columns in P_o, as shown in theorems 4 and 5. For instance, one circle can be inserted to P_n if P_o contains 8 rows and 5 columns and for this given number of rows two circles can be inserted to P_n if a column is added to P_o.

However, since the number of rows to be added to P is the same for the interval ,
. n ..464646)1] where x is any
integer, then repositioning the rows in excess of $[\, x(7.464) \,]$ but less than $[\, 7.464 \,]$ result in the displacement of some circles because the alternate rows contain fewer circles.

Thus, in order to obtain the maximum number (T_m) of circles that can be fitted to a new pack (P_m) it is logical to reposition the number of rows equal to $[\, x(7.464) \,]$ and to retain in their original positions the number of excess rows less than $[\, 7.464 \,]$. This situation is illustrated in figures 7(a), 7(b) and 7(c), where P_o contains 50 circles with 10 rows and 5 columns.

The procedure of computing the maximum number (T_m) of circles that can be fitted to P_m is given in theorem 6.

Theorem 6. Let x be the number of rows that can be added to P_n, r and c be the number of rows and columns in P_o and T_x be a subset of r that can allow the addition of x rows. Then the number (T_m) of circles that can be fitted to a new pack (P_m) is maximum if the T_x rows are the only ones repositioned. T_m is given below:

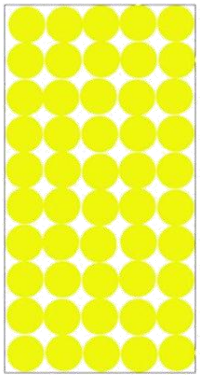
$T_m = ½[2c(x+r)–(Tx+x)]$, if T_x and x are of the same parity
$= ½ [2c (x + r) – (Tx + x) + 1]$, if T_x and x are of different parity

Where, as before, $x = \frac{r}{7.464}$ and

$T_x = [\, x(7.464) \,]$.

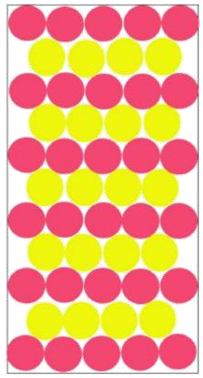
Proof:

By assumption, T_x is the number of rows in r that can allow the addition of x more rows after repositioning and thus $r–T_x$ is the number of rows in r that cannot allow the addition of a row by the



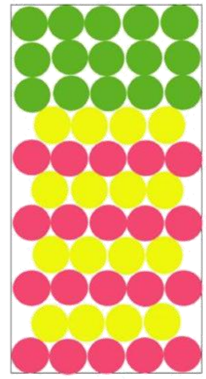
P_o:rc

Figure 7(a)



P_a:T_a

Figure 7(b)



P_m:T_m

Figure 7(c)

same action. The total number of circles in T_x rows is given in Theorem 3, which is either ½[(r + x)(2c-1)] or ½ [(r + x)(2c – 1) +1], depending on whether x and r are of the same or different parity, and where r is replaced by T_x. The total number of circles in r–T_x rows is (r–T_x) c because this part is made to remain in its original position. Thus, the total number (T_m) of circles that can be fitted in P_n by partial repositioning is

$T_m = T_n + (r - T_x)c$.
Where: $T_n = ½(Tx + x)(2c-1)$ or
$= ½ [(Tx + x)(2c - 1) + 1]$

Case 1. When x and T_x are of the same parity.

$T_m = T_n + (r - T_x) c$
$= ½(Tx + x)(2c-1) + (r - Tx)$ c, by Theorem 3.
$= ½ [2c(x + r) - (Tx + x)]$ on solving

Case 2. When x and T_x are of different parity

$T_m = T_n + (r - T_x) c$
$= ½ [(Tx + x)(2c - 1) + 1] + (r - Tx)$, by Theorem 3.
$= ½ [2c(x + r) - (Tx + x) + 1]$ on solving

It now remains to show that $T_m \geq T_n$ for any column (c). It suffices to consider only one case, say, case 1.

Show that $T_m \geq T_n$. Assume that r and x are of the same parity. Then

$T_n = ½(r + x)(2c-1)$
$= ½[(T_x + x) + (r - T_x)](2c-1)$, since $r = T_x + (r - Tx)$
$= ½[(T_x + x)(2c-1) + (r - T_x)(2c-1)]$
$\leq ½[(T_x + x)(2c-1) + (r - T_x) 2c]$, since $(r - T_x) \geq 0$
$= ½[(2cT_x + 2cx - T_x - x + 2cr - 2cT_x] = ½[(+ 2cx + 2cr - T_x - x] = ½[2c (x + r) - (T_x + x)]$
$= T_m$

Therefore T_m is the maximum number of circles that can be fitted in P_m.

Corollary 1: $T_m > P_o$, if $x \geq 1$ and $c \geq 5$.
Corollary 2: $T_m = T_n$, if r is a multiple of Tx.
Corollary 3: $T_m \geq T_n$, if r is not a multiple of T_x and $x > 0$.

Maximum Number of Circular objects that can be Inserted in P_m

The formula in determining the number (T_c) of circles that can be inserted in P_m is derived similarly as the formula in determining the number of circles that can be inserted in P_n. Thus, if x is the number of rows that can be added, T_x is the number of rows that can allow the insertion of x additional rows and r is the number of rows in P_o, then,

$T_c = ½ [x (2c-1) - T_x]$, if x and T_x are of the same parity ½ $[x (2c - 1) + (1 - Tx)]$, if x and T_x are of different parity Where $x = \frac{r}{7.464}$, as before and

$$T_x = \lceil x(7.464) \rceil$$

As illustrations, compute the maximum number of circular objects that can be fitted in the default pack, P_o, new pack, P_n and new pack, P_m, given that the rectangular container (or RC space) has dimensions, r x c = 20 x 10.

Solutions

1. The number of circular objects that can be fitted in P_o is T_o.

$$T_o = rc = (20)(10) = 200$$

2. The number of objects in P_n is T_n.

$$x = \frac{r}{7.464} = \frac{20}{7.464} = [\,2.680\,] = 2$$

$T_n = ½(r + x)(2c-1)$, since r and x are of the same parity

$= ½(20+2)[2(10)-1]$
$= ½(22)(19)$
$= 209$

14 15

3. The number of objects in P_m is T_m.

$T_x = [\,x(7.464)\,] = [\,2(7.464)\,] = [\,14.928\,] = 15$, since x = 2

20

$T_m = ½\,[2c\,(x + r) – (T_x + x) + 1]$, Since x and T_x are of different parity

$= ½\,[(2)(10)\,(2 +20) – (15 +2) + 1]$
$= 212$

4. $T_c = ½\,[x\,(2c – 1) + (1 - T_x)]$, since x and T_x are of different parity: x = 2, T_x = 15.

$= ½\,[2\,(2(10) – 1) + (1 - 15)]$
$= ½\,[38) + (-14)]$
$= ½\,(24)$
$= 12$

In the given examples, P_n contains 9 objects more than that of P_o and in turn, P_m contains 3 objects more than that of P_n. Therefore, packing a rectangular container with circular objects by way of P_m yields the maximum content of the container.

Refinement of the Models

A refinement of the above models of computing the maximum number of circular objects that can be contained in P_n, or the maximum number of objects that can be added to P_n, due to repositioning may be made by simply utilizing the fact about the minimum number of rows, r, in P_o that allows the addition of one more row in P_n and about the minimum number of columns, c, in P_o that allows the addition of an object in P_n.

Since addition of a row in P_n is possible every time the number of rows in P_o is a multiple of 7.464 and addition of an object is possible whenever the number of columns, c, in P_o is $c \geq 5$, then the total number, T_a, that can be added to P_n, and consequently the total number, T_n, of objects that can be contained in P_n are the following:

$$T_a = (c-4)\left(\frac{r}{7.464}\right)$$

or if $x = \dfrac{r}{7.464}$

$\mathbf{T_a = (c-4)x}$ and

$T_n = rc + T_a$
$= rc + (c-4)x$
$= rc + cx - 4x$

$$\mathbf{T_n = c(r+x) – 4x}$$

From the model, it should be noted that T_n contains maximum number if r is equal to $[\,k(7.464)\,]$ for and $c \geq 5$.

CONCLUSIONS AND RECOMMENDATIONS

Based on from the results, this study shows that there are deterministic mathematical models of calculating the maximum number of identical circular objects that can be packed into rectangular spaces.

The models developed may be used to determine the maximum number of circular objects that can be packed in a given rectangular space, be that objects be vials in hospital, cigarette sticks in factories, pipes in hardware or planting materials in agriculture.

The mathematical models generated from the study may be used as bases in designing a rectangular container that can accommodate the maximum number of objects to be packed.

The mathematical procedures generated from the study may be a source of academic discourses in the academe, particularly serving as an example of a mathematical investigatory undertaking.

Since the study does not consider the situation where the rectangular space provides extra space along its dimensions in which case, necessary adjustment on the existing model has to be made, it is recommended that future study may consider such a case.

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