THE MATHEMATICS OF MAXIMIZING A PARALLELEPIPED BOX WITH SPHERICAL OBJECTS

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ABSTRACT

This study investigated a special case of packing problems involving identical and fixed sized spherical objects and rectangular parallelepiped box. It determined the possible patterns of piling spherical objects into a parallelepiped box so that the box attains its maximum content. The study explored on the different ways of piling and identified those that yield content greater than that in ordinary piling. The effects of the dimensions of a box on whether or not it is possible to obtain a new pile that tends to increase the population density of the box is determined. The results show that there are smooth and behaved piling patterns that tend to increase the population density of the box from its default piling pattern. It is found that if the box can contain at least 5 spheres along its length, at least 3 spheres along its width and at least 4 spheres along its height, then it is possible to modify the default pile pattern so that additional spheres can be fitted into the box. The mathematics of maximizing the content of the box, or by increasing its population density, is given in theorems and corollaries. Also, the proofs of the theorems are supplied to establish their mathematical viability. The results also show that it is possible to generate mathematical models that establish deterministic algorithms of maximizing content, or increasing population density of the parallelepiped box. The mathematical models developed are recommended for use in calculating the maximal content of a given parallelepiped container. Finally, it is recommended that further investigations on the same topic be conducted as the present study is in no way exhaustive.

Key words***: Parallelepiped box, maximizing space, spherical objects, and mathematical piling models.***

INTRODUCTION

With the growing recognition by educators worldwide of a learning philosophy (constructivism) and theories (*e.g.* Piaget and Bruner) that place student at the center of the educational processes, problem solving and mathematical investigation are now being placed at the core of mathematics learning. According to Yeo (2009), many school mathematics curricula such as those in Australia, New Zealand and the United Kingdom emphasize the use of problem solving and mathematical investigation in the teaching and learning of mathematics.

What is mathematical investigation? A mathematical investigation is defined as a “collection of worthwhile problem solving tasks that has multidimensional content; is open-ended, permitting several acceptable solutions; is an exploration requiring a full period or several classes to complete; is centered on a theme or event; and is often embedded in a focus question.

In addition, a mathematical investigation involves a number of processes, which include researching outside sources to gather information; collecting data through such means as surveying, observing, or measuring; collaborating, with each team member taking on specific jobs; and using multiple strategies for reaching solutions and conclusions” (www. beenleigss.eq.edu.au).

According to Yeo (2009), some educators viewed mathematical investigation as “an open-ended problem or statement that lends itself to the possibility of multiple mathematical pathways being explored, leading to a variety of mathematical ideas and/or solutions” (Baily, 2007); as an “open problem with open goal” (Orton and Frobisher, 1996) and “open answer” (Pirie, 1987); as involving both “problem posing and problem solving” (Cai and Cifarelli, 2005); and, among others, as the “process” (Evans, 1987) involved in the investigative task. An open problem or investigative task “allows students to choose what goal to pursue and an open answer allows them to have many correct answers" (Yeo and Yeap, nd).

The benefits of mathematical investigation as an approach to teaching mathematics, according to some educators as cited by Yeo (2009), include a) "getting students more interested" (Davies, 1980) to learn and "more open to work mathematically" (Tanner, 1989). On the part of the learner, doing mathematical investigation, according to some researchers as cited by Nievera (2012), "deepening their understanding of the content of mathematics and challenges them to produce their own mathematics within their universe of knowledge" as well as "developing their thinking processes and good mental habbit" (Orton and Frobisher, 2005).

The subject of this investigation, which is maximizing a parallelepiped box with spherical objects, is one of the many circle and sphere packing problems in operation research that have been explored by many researchers worldwide using various sophisticated strategies such as the development of "algorithms ranging from computer – aided optimality proofs, to branch-and-bound procedures, to constructive approaches, to multi-start nonconvex minimization, to multiphase heuristics, and metaheuristics" (Hifi and M'Halla, 2009). Among the studies conducted on packing problems involving circular and spherical objects include that of Liu, et al. (2009) on "An Effective Hybrid Algorithm for the Circles and Spheres Packing Problems", Molnar, (1978) on "Packing of Congruent Spheres in a Strip", Birgin, (2008) on "Minimizing the object dimensions in circle and sphere packing problems", Wang, (1999) on "Packing of unequal spheres and automated radio surgical treatment planning", Gensane, (2004) on "Dense packing of equal spheres in a cube", Stoyan, and Yaskov, (2008) on "Packing identical spheres into a rectangular parallelepiped", Stoyan, *et al*. (2003) on "Packing of various radii solid spheres into a parallelepiped", and Sutou, and Dai, (2002) on " Global optimization approach to unequal sphere packing problems in 3D".

The foci of the strategies developed in the above studies have been on determining the optimal densities of packing both 2D and 3D rectangular and circular spaces with circular and spherical objects wherein either the radii of the objects are minimized to achieve optimal cover of a given space or the dimensions of the containing spaces are minimized. Although the authors described in great details how their strategies operate for some specified number of objects to be contained or for some given dimensions of containing spaces, the implementations of such strategies were not shown in the published papers.

In this work, the special case of packing problem involving spheres is considered in which the spheres to be packed are identical and fixed size and the unit of measure of the dimensions of the parallelepiped container is the diameter of one sphere. In order to develop some deterministic formulas of manually computing the maximal content of a given parallelepiped container, only the more behaved positioning patterns that have potentials of increasing the packing densities are considered. In contrast, the above mentioned studies deal with random scattering of both identical and varied sized objects in the containing spaces, which is the general case in packing problems.

This simple investigatory work explored the possible ways of populating a given rectangular parallelepiped box with identical spherical objects so that the box will attain its maximal content. Figure 1 shows the default position of the spheres in the parallelepiped box in which the spheres are positioned along straight lines both vertically and horizontally.

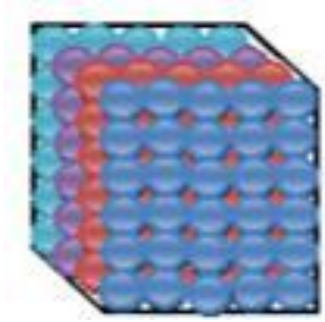

Figure 1. Default pile of spheres in the paralleleppiped box

If the dimension of the box is measured in terms of the number of spheres that can be fitted along its sides, or that the unit of measure of the box is one sphere, then the total number (V_o) of spheres in the box when fully filled and arranged in default position, as in figure 1, is determined using the typical formula of the volume of a parallelepiped box below:

$V_o = l\ w\ h$ Where: l = number of spheres at the length of the box
w = number of spheres at the width
h = number of spheres at the height.

The above rectangular, or default position, of the spheres is called the old pile, P_o. If the default position is completely altered such that no two consecutive layers of the pile remain in their original position then the resulting arrangement shall be called the new pile, P_n. Now, is it possible to obtain a P_n that contains more spheres than P_o?

Research Questions

1. What is the minimum dimension of the parallelepiped box that permits the generation of a new pile, P_n, which contains more spheres than the default pile, P_o?

2. What are the forms of the new piles that can maximize the space in a box?

3. What are the dimensions of the pile that can contain the least or the most number of spheres?

3. What are the mathematical models representing the volumes of the new piles?

METHODOLOGY

The study is exploratory in nature. The default pile (P_o), which is a pile in rectangular form, is altered in any form inside the parallelepiped box to produce a new pile. Different forms of piling arrangements were explored on different dimensions of the parallelepiped boxes. The effects of the different forms of piling on the vertical heights of the piles were calculated and compared to determine which form of piling yields the greatest number of spheres. Also, the effects of the dimensions of a box on the amount of spheres to be fitted in the different piles were calculated to determine which form of piling is most dense or least dense for a particular range of dimensions. The mathematical models of the different forms of piling were developed and their proofs constructed to show their mathematical viability.

RESULTS AND DISCUSSION

The exploration of the topic starts with the given conjecture below.

Conjecture

If the old pile, P_o, is altered as described above, then the new pile, P_n, may contain a number of spheres less than, equal to, or greater than that of the old pile, P_o.

Proof of the Conjecture

The proof of the conjecture may be given in the proofs of the succeeding theorems.

The possibility of P_n to contain more number of spheres than P_o by any complete alteration of P_o may happen only if the said alteration will result in the creation of space that will allow the addition of at least one layer of spheres in P_n. The nature of alteration that has the net effect of increasing the number of

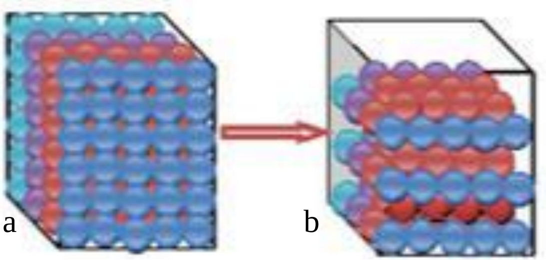

Figure 2 (a) Default position os spheres
2 (b) Altered position of spheres

layers to be contained in the new pile is to arrange the spheres in pyramidal formations by positioning each sphere in the alternate layers (or even number layers) at the middle of any four spheres in the lower layers (or odd number layers) as in figures 2(a) and 2(b) below.

Based on the alteration shown in figure 2(b), the relevant question to be answered is how many layers of spheres in P_o will allow the addition of one more layer in P_n?

Theorem 1 gives the minimum number of layers in P_o that can allow the addition of one more layer in P_n.

Theorem 1: Let Po be altered in which each piece of sphere in the alternate layers (or even number layers) is positioned at the middle of any four pieces of sphere in the lower layers (or odd number layers). Let this form of alteration be called A_n alteration. Then the minimum number, n, of layers in P_o that can allow the addition of one more layer in P_n by A_n alteration is 4.

Proof

If A_n alteration is performed, then the vertical distance between the centers of the spheres across any two

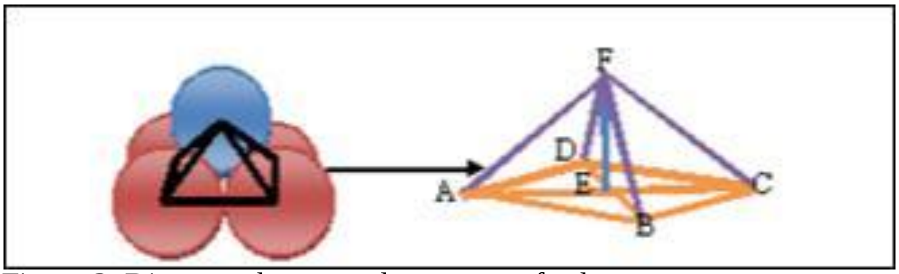

Figure 3. Distances between the centers of spheres

layers in P_o becomes the slant height in P_n. Moreover, when the centers of any five spheres across two consecutive layers are joined together as in figure 3 below, they form an equilateral pyramid having a square base.

Since Pyramid ABCDF is equilateral, AB = BC = CD = AD = AF = DF = BF = CF = d (or the diameter of a sphere). Also, since the base ABCD is a square, the length of line segment AC is computed using Pythagorean Theorem as follows:

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

$$\overline{AC}^2 = d^2 + d^2 = 2d^2$$

$$\overline{AC} = \sqrt{2}d,$$

But $\overline{AC} = 2\overline{AE}$,

which yields $\overline{AE} = \frac{\overline{AC}}{2} = \frac{\sqrt{2}}{2}d$

Also, triangle AEF forms a right triangle with the right angle at E. Hence, the distance, $\overline{EF}$, between the centers of the spheres across two consecutive layers in Pn, is

$$\overline{EF}^2 = \overline{AF}^2 - \overline{AE}^2 = d^2 - (\frac{\sqrt{2}}{2}d)^2 = \frac{d^2}{2}$$

$$\overline{EF} = \frac{\sqrt{2}}{2}d$$

Since $\overline{EF} = \frac{\sqrt{2}}{2}d$ <d, then it shows that the alteration of the positions of the spheres in P_n creates a height space that may accumulate to allow the addition of a layer. In order to determine the minimum number, n, of layers in P_o that permits the addition of a layer in P_n, it is natural to think that such is possible only if the difference between the height, h, of P_o and ■ of P is equal to the $\frac{\sqrt{2}}{2}d$ $_n$ of a layer P_n, which is the value of the new distance between the centers of spheres across layers.

Thus,

$$h - h_n = \frac{\sqrt{2}}{2} d$$

In terms of the diameter, d, of the sphere, the number of layers, h_n , in P_n and h of P_o are

$h_n = (n-1)\frac{\sqrt{2}}{2}d + d$ and $h = nd$, respectively.

Thus,

$$h - h_n = \frac{\sqrt{2}}{2} d$$

$$nd - \left(n\frac{\sqrt{2}}{2}d - \frac{\sqrt{2}}{2}d + d \right) = \frac{\sqrt{2}}{2}d$$

$$[2nd - n\sqrt{2}d + \sqrt{2}d - 2d] = \sqrt{2}d$$

$$2nd - n\sqrt{2}d = 2d$$

$$n = 2d/(2d - \sqrt{2}d)$$
$$= 2/(2 - \sqrt{2})$$
$$= 3.4142 \approx 4$$

as desired.

Corollary 1: Let y be the number of layers that can be added to P_n by way of A_n alteration. Then the value of y for any given value of h is

$$y = \frac{h}{3.4142}$$

Which is the greatest integer less than or equal to $\frac{h}{3.4142}$.

Corollary 2: The value of y is the same for the interval $[y(3.4142)] \leq k < [(y+1)(3.4142)]$ where k is any specific number of rows in the interval.

Illustrative Example

How many layers of spheres can be added to the new pile, P_n, given the following layers of P_o: 11, 13 and 14? How many layers of spheres are in P_n?

Solutions

a. $h=11$

$$y = \frac{11}{3.4142} = [3.2218] = 3$$

The number of layers, h_n, of P_n is h plus y. In symbol

$$h_n = h + y = 12 + 3 = 15.$$

b. h=3

$$y = \left[\frac{13}{3.4142}\right] = [3.8076] = 3$$

$$h_n = 13 + 3 = 16$$

c. h = 14

$$y = \left[\frac{14}{3.4142}\right] = [4.1005] = 4$$

Thus the value of y is the same for the different values of h within the interval [11, 13].

The next problem to resolve is to determine a model for computing the volume of P_n whenever the dimension of P_o is given. Theorem 2 gives the formula for the volume of P_n.

Theorem 2: Let V_o be the volume of P_o with dimensions, lwh, where l is the length, w is the width and h is the height. Also, let y be the number of layers to be added to P_n due to A_n alteration. If V_n is the volume of P_n and the unit of measure is one sphere, then V_n is given below.

V_n= 1/2 (h + y) (2lw-l-w +1) if h and y are the same parity
= 1/2 [2lw(h+y) - (l+w-1)(h+y-1)], if h and y are of different parity

where $y = \dfrac{h}{3.4142}$

Proofs

Case 1:Both h and y are of the same parity

a) Both h and y are odd numbers

If h is odd then the number of odd layers in his $\frac{(h+1)}{2}$ and that of the even layers is $\frac{(h-1)}{2}$. Since the odd layers are to remain unaltered and the even layers are the ones to be altered, then the dimension of the odd layers is while that of the even layers is (l-1)(w-1). Also, if h is odd, the last layer in h has a dimension of lw and thus the first layer in y has a dimension of (l-1)(w-1). Thus

$$V_n = \left(\frac{h+1}{2}\right)lw + \left(\frac{h-1}{2}\right)(l-1)(w-1) + \left(\frac{y+1}{2}\right)(l-1)(w-1) + \left(\frac{y-1}{2}\right)lw$$

$$= 1/2[(h+1)lw + (h-1)(l-1)(w-1) + (y+1)(l-1)(w-1) + (y-1)lw]$$

$$= 1/2(lwh + lw + lwh - wh - lh + h - lw + w + l - 1 + lwy$$
$$- wy - ly + y + lw - w - l + 1 + lwy - lw)$$

$$= 1/2(2lwh + 2lwy - wh - wy - lh - ly + h + y)$$

$$= 1/2[(2lw(h+y) - h(w+l-1) - y(w+l-1)]$$

$$= 1/2[2lw(h+y) - (l+w-1)(h+y)]$$

$$= 1/2\,(h+y)(2lw - l - w + 1)$$

b) Both h and y are even numbers

If both h and y are even numbers then the number of odd and even layers in both h and y is equally divided, which are $\frac{h}{2}$ for both odd and even layers in h and $\frac{y}{2}$ for both odd and even layers in y. Thus

$$V_n = \frac{h}{2}lw + \frac{h}{2}(l-1)(w-1) + \frac{y}{2}lw + \frac{y}{2}(l-1)(w-1)$$
$$= 1/2(lwh + lwh - wh - lh + h + lwy + lwy - wy - ly + y)$$
$$= 1/2[2lwh + 2lwy - h(l + w - 1) - y(l + w - 1)$$
$$= 1/2(2lw(h + y) - (l + w - 1)(h + y)]$$
$$= 1/2\ (h + y)(2lw - l - w + 1)$$

Case 2: h and y are of different parity

a. h is odd number and y is even number

If h is an odd number, the number of layers in h that is to remain unaltered is $\frac{(h+1)}{2}$ and that to be altered is $\frac{(h-1)}{2}$. Since y is even, the number of layers in y that is to remain unaltered and that to be altered are the same whi ~~ch~~ is $\frac{y}{2}$. Thus

$$V_n = (\frac{h+1}{2})lw + (\frac{h-1}{2})(l-1)(w-1) + \frac{y}{2}lw + \frac{y}{2}(l-1)(w-1)$$
$$= 1/2(lwh + lw + lwh - wh - lh + h - lw + w + l - 1 + lwy - wy - ly + y)$$
$$= 1/2[(2lw(h + y) - (h + y - 1)(l + w$$

$-1)$] b. h is even and y is odd

If h is even, the number of layers to remain unaltered and that to be altered is both $\frac{h}{2}$; while for y the number of layers in it that is to be altered is $\frac{(y + 1)}{2}$ and that to remain unaltered is $\frac{(y - 1)}{2}$. Hence,

$$V_n = \frac{h}{2}lw + (\frac{h}{2})(l - 1)(w - 1) + (\frac{y - 1}{2})(l - 1)(w - 1)$$
$$= 1/2[lwh + lwh - lh - wh + h + lwy + lw + lwy - ly - wy + y - lw + l + w - 1)]$$
$$= 1/2[2lw(h + y) - l(h + y - 1) - w (h + y - 1) + (h + y - 1)]$$
$$= 1/2[2lw(h + y) - (h + y - 1) (l + w - 1)]$$

Illustrative Example

How many spheres can be accomodated in a parallelepiped box with a default dimension of lwh =5x6x7?

Solutions: find for V_o and V_n

a. $V_o = lwh$ = 5x6x7 = 210

b. V_n, given h = 7

$$\frac{7}{3.4142} = [\ 2.0502\ ]$$

Since h is odd and y is even, the problem falls under case 2(a).
Thus the number of spheres in P_n is

$$V_n = 1/2[(2lw(h+y) - (h+y-1)(l+w-1)]$$
$$= 1/2[(2x5x6(7+2) - (7+2-1)(5+6-1)]$$
$$= 1/2\,(540 - 8(11))$$
$$= 1/2\,(452)$$
$$= 226$$

The volume, V_n, of the new pile, P_n, is greater than the original volume, V_o, of the old pile, P_o, because as earlier stated performing the alteration, A_n, creates space that allow the addition of some more spheres into the original pile.

The next problem to resolve is to determine the minimum dimension of P_o that allows the addition of one sphere to P_n. It is noted that the increase in the number of layers in P_n by A_n alteration results in the displacement of some spheres in the layers altered, thus making it difficult to say whether the alteration will result in increase or decrease of the content of P_n. If the number of spheres displaced is equal to the number of spheres in the layer added, then V_o of P_o contains as much as V_n of P_n. If the number of spheres displaced is more than that of the layers added, then V_n contains lesser number of spheres than V_o. Finally, if the number of spheres displaced is less than that of the layers added, then V_n is greater than V_o.

Theorem 3 gives the minimum dimensions of P_o in which it is possible to add a sphere to P_n by A_n alteration.

Theorem 3: Let the dimension of P_o be lwh, where l is the length, w the width and h the height. If one sphere is the unit of measure, then the minimum dimension of P_o that will allow the addition of a sphere in P_n by A_n alteration is as given below

$$\text{Dim}P_o = lwh = 5 \text{ x } 3 \text{ x } 4$$

Proof

If V_n is made to exceed V_o by one sphere by way of A_n alteration, then the equation below is in order

$$V_n - V_o = 1$$

Given $h = 4$ as the minimum number of layers in P_o that will allow the addition of a layer ($y = 1$) in P_n, what remains to be determined are the minimum length, l, and minimum width, w, of P_o that will make V_n exceed V_o by one sphere.

Since $h = 4$ and $y = 1$, the situation falls under case 2 of Theorem 2 and thus the appropriate formula to determine V_n is

$$V_n = 1/2[2lw(h+y) - (h+y-1)(l+w-1)]$$

Thus

$$V_n - V_o = 1 \quad \Leftrightarrow 1/2[2lw(h+y) - (h+y-1)(l+w-1)] - lwh = 1$$

$$\Leftrightarrow 1/2[2lw(4+1) - (4+1-1)(l+w-1)] - lw(4) = 1$$

$$\Leftrightarrow 1/2[10lw - 4(l+w-1)] - 4lw = 1$$

$$\Leftrightarrow 1/2(10lw - 4l - 4w + 4) - 4lw = 1$$

$$\Leftrightarrow (5lw - 2l - 2w + 2) - 4lw = 1$$

$$\Leftrightarrow lw - 2l - 2w + 2 = 1$$

$$\Leftrightarrow l(w-2) = 2w - 1$$

$$\Leftrightarrow l = \frac{2w-1}{w-2}$$

Since $l \in Z$ and $w - 2 \neq 0$, the least value that w can assume is 3. So if $w = 3$ then the value of l is

$$l = \frac{2w-1}{w-2} = \frac{2(3)-1}{3-2} = 5$$

Therefore, the minimum dimensions of P_o that allow the addition of one sphere in P_n by A_n alteration is lwh = 5x3x4, as desired.

Illustrative example

1. Let the dimension of P_o be the required minimum of lwh = 5x3x4. Verify that it is possible to insert a sphere into the pile.

Solution: Find V_o and V_n

a. $V_o = lwh = 5\text{x}3\text{x}4 = 60$

b. V_n, given that $h = 4$

$$y = \left[\frac{h}{4}\right] = \left[\frac{4}{4}\right] = [1] = 1$$

Because $h = 4$ and $y = 1$ are of different parity, the problem falls under case 2. So the volume, V_n, of the new pile is

$$V_n = 1/2[2lw(h+y) - (h+y-1)(l+w-1)]$$
$$= 1/2[2(5)(3)(4+1) - (4+1-1)(5+3-1)]$$
$$= 1/2(150 - 28)$$
$$= 1/2(122)$$
$$= 61$$

$$V_n - V_o = 61 - 60 = 1$$

Thus if the default dimension of the pile is lwh = 5x3x4, then a sphere can be inserted by performing A_n alteration.

2. Suppose the dimension of P_o is lwh = 6x6x3, how many spheres can be contained in

P_n? Solution: Find V_o and V_n.

a. $V_o = lwh = 6x6x3 = 108$
b. V_n, given $h = 3$

$$y = \frac{h}{4} = \frac{3}{4} = \left[0.75 \right] = 0$$

Since h = 3 and y = 0 are of different parity, then the problem falls under case 2. Thus the volume, V_n, of the new pile is

$$V_n = 1/2[2lw(h + y) - (h + y - 1)(l + w - 1)]$$
$$= 1/2[2(6)(6)(3 + 0) - (3 + 0 - 1)(6 + 6 - 1)]$$
$$= 97$$

There are 11 spheres displaced by A_n alteration, but there is no layer added because h=3<4 (which is the minimum value of h to allow the addition of a layer). Thus $V_n < V_o$.

3. Given that P_o has the dimension lwh = 4x3x11, find the number of spheres in P_n.

Solution: $l = 4$, $w = 3$, $h = 11$

$$y = \frac{h}{3.4142} = \frac{11}{3.4142} = \left[3.22 \right] = 3$$

h and y are of the same parity, so the problems falls under case 1. Hence

$$V_n = 1/2(h + y)(2lw - l - w + 1)$$
$$= 1/2\,(11 + 3)[(2)(4)(3) - 4 - 3 + 1)]$$
$$= 126$$

Whereas $V_o = lwh$ = 4x3x11 = 132.

V_n contains less number of spheres than V_o because the value of l is less than the required minimum value of 5, even though the value of the other variables are way above the minimum.

c. Find the content of P_n, given the dimension of P_o to be lwh = 6x3x4

Solution: $l = 6$, $w = 3$, $h = 4$

$$y = \frac{4}{3.4142} = \left[1.17 \right] = 1$$

h and y are of different parity, so the problem falls under case 2. Hence V_n is

$$V_n = 1/2[2lwh(h+y) - (h=y - 1)(l + w-1]$$

$$V_n = 1/2[2(6)(3)(4+1) - (4+1-1)(6+3-1)]$$

$$= 74$$

Whereas V_o is

$$V_o = lwh = 6x3x4 = 72$$

V_n is greater than V_o because the dimension of P_o exceeds the required minimum of 5x3x4.

Rather than solving for the value of both V_o and V_n to know whether or not V_n contains less than, equal to, or greater than that of V_o, it is much more efficient to accomplish the same objective by plugging in the data to the function that governs the difference between the two volumes ($V_n – V_o$). Theorem 4, gives the form of the said function of difference.

Theorem 4: Let D_n be the number of spheres to be displaced from or added to V_n due to A_n alteration,

$$\text{then } D_n = \begin{cases} 1/2[2lwy - (h+y)(l+w-1)], & \text{if } h \,\&\, y \text{ are of the same parity} \\ 1/2[2lwy - (h+y-1)(l+w-1)], & h \,\&\, y \text{ are of different parity} \end{cases}$$

$$\text{where } y = \frac{h}{3.4142}$$

Proof

The number of spheres to be displaced from or added to V_n is that yielded when V_o is subtracted from V_n. Thus

$$D_n = V_n - V_o$$

Case 1: h and y are of the same parity

$$D_n = 1/2(h+y)(2lw - l - w + 1) - V_o$$

$$= 1/2[(h+y)(2lw - l - w + 1) - 2V_o]$$

$$= 1/2[(h+y)(2lw) - (h+y)(l+w-1) - 2lwh]$$

$$= 1/2[2lwy - (h+y)(l+w-1)]$$

Case 2: h and y are of different parity

$$D_n = 1/2[2lw(h+y) - (l+w-1)(h+y-1)] - Vo$$

$$= 1/2[2lwh + 2lwy - (l+w-1)(h+y-1) - 2lwh]$$

$$= 1/2[2lwy - (l+w-1)(h+y-1)]$$

Corollary 2: A zero value of D_n means $V_o = V_n$
Corollary 3: A positive value of D_n means $V_o < V_n$
Corollary 4: A negative value of D_n means $V_o > V_n$

Proofs

Note that $D_n = V_n – V_o$.

If $D_n = 0$ $\quad 0 = V_n - V_o \quad V_0 = V_n$.
If $D < 0 \Rightarrow V - V < 0 \Rightarrow \quad V < V$.
If $D^n > 0 \Rightarrow V^n - V^o > 0 \Rightarrow V^n > V .^o$
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Illustrative example

For purpose of illustration, it is enough to deal with the 4th corollary. The other two corollaries can be easily verified from the previous examples. Let P_o has a dimension of lwh = 5x3x2. Find D_n.

Solution: Solve first for y and then for D_n.

$$y = \frac{h}{3.4142} = \frac{112}{3.4142} = [[33.220..225858]=3] = 3_0$$

Since h and y are of the same parity, the problem falls under case 1. Thus

$$= 1/2[2(5)(3)(0) - (2 + 0)(5 + 3 - 1)]$$
$$= 1/2\,(0 - 14)$$
$$= -7$$

Because the value of Dn is negative, it means that $V_o > V_n$. In fact, the old pile, P_o, contains 7 spheres more than that of the new pile, P_n.

Theorem 5: If the dimension of P_o is less than the minimum dimension of lwh = 5x3x4,then $V_n \leq$

V_o. **Proof**

The dimension of P_o is less than the minimum dimension of 5x3x4 if the dimension of a component is less than the minimum; that is, if $l < 5$, or $w < 3$, or $h < 4$.

Case 1: $l < 5$, $w = 3$, $h = 4$

If l< 5, then l can take any value from 2 to 4. It suffices to let l takes the highest value of 4. Moreover, the value of y is

$$y = \frac{4}{3.4142} = [\,1.1716\,] = 1$$

Since h and y are of different parity, then

$$D_n = 1/2[2lwy - (l + w - 1)(h + y - 1)]$$
$$= 1/2[2(4)(3)(1) - (4 + 3 - 1)(4 + 1 - 1)]$$
$$= 0$$

This means that $V_n = V_o$. Consequently $P_n = P_o$

Case 2: $w < 3$, $l = 5$, $h = 4$

The only possible value that w can take if it is less than 3 is 2. As before

$$y = \frac{4}{3.4142} = [1.1716] = 1$$

Since h and y are of different parity, the problem falls under case 2 for D_n. Thus

$$D_n = 1/2[2lwy - (l + w - 1)(h + y - 1)]$$
$$= 1/2[2(5)(2)(1) - (5 + 2 - 1)(4 + 1 - 1)]$$
$$= -2$$

This means that V_n contains 2 spheres less than that of V_o. Thus $P_n < P_o$.

Case 3: $h < 4$, $l = 5$, $w = 3$

The allowable values that h can take are 2 and 3. Let $h = 3$. Moreover, in order to determine which case of D_n is to be considered, the value of y is solved.

$$y = \frac{3}{3.4142} = [0.88] = 0$$

Since h and y are of different parity, the value of D_n is computed using

$$D_n = 1/2[2lwy - (l + w - 1)(h + y - 1)]$$
$$= 1/2[2(5)(3)(0) - (5 + 3 - 1)(3 + 0 - 1)]$$
$$= -7$$

The result means that V_n contains 7 spheres less than that of V_o. So $P_n < P_o$.

In all cases $P_n \leq P_o$.

From the foregoing results, it is seen that the given form of piling maximizes a parallelepiped space whenever the dimensions of the space meet the required minimum of lwh = 5x3x4; otherwise, the pile minimizes the content of the space.

It should be noted that, by the manner Pn pile is formed in which the even number layers have dimensions of (l-1)(w-1) in exchange of achieving vertical compression so that extra layers can be added, the volume, V_n, of P_n is minimal if the value of h lies in between two consecutive multiples of 3.4142 .This assertion is given in theorem 6.

Theorem 6. Let h be, as usual, the number of L W layers in P_o. Then V_n of P_n attains its maximal value when $h = [y(3.4142)]$ and attains it minimal values when $[y(3.4142)] < k < [(y + 1)(3.4142)]$.

Proof

Given two values of h, which are $h_1 = [y(3.4142)]$ and $h_2 = [y(3.4142)] + k$, where $k \leq 3$. Since $h1$ is a multiple of $[y(3.4142)]$ and h_2 lies in between $[y(3.4142)]$ and $[(y + 1)(3.4142)]$, it is enough to show that the number of spheres to be added to P_n for h_1 is greater than or equal to that for h_2. Thus if D_{n1} is the number of spheres to be added to P_n for h_1 and D_{n2} is the number of spheres to be added to P_n for h_2, it is enough to show that $D_{n1} - D_{n2} \geq 0$ for k =1, 2, and 3.

Without loss of generality, assume, initially, that both h1 and y are even. Then h2 is odd. Therefore

$$D_{n1} = 1/2[2lwy - (h_1 + y)(l + w - 1)]$$
$$D_{n2} = 1/2[2lwy - (h_2 + y - 1)(l + w - 1)]$$
$$D_{n1} - D_{n2} = 1/2[2lwy - (h_1 + y)(l + w - 1)] - 1/2[2lwy - (h_2 + y - 1)(l + w - 10]$$

Now, let k = 1. Then $h_2 = h_1 + 1$, and

$$\begin{aligned} D_{n1} - D_{n2} &= 1/2[\{2lwy - (h_1 + y)(l + w - 1)\} - \{2lwy - (h_1 + 1 + y - 1)(l + w - 1)\}] \\ &= 1/2[\{2lwy - (h_1 + y)(l + w - 1)\} - \{2lwy - (h_1 + y)(l + w - 1)\}] \\ &= 0 \end{aligned}$$

$$\begin{aligned} &= 1/2[\{2lwy - (h_1 + y)(l + w - 1)\} - \{2lwy - (h_1 + y + 2)(l + w - 1)\}] \\ &= 1/2[-(h_1 + y)(l + w - 1) + (h_1 + y + 1)(l + w - 1)] \\ &= (l + w - 1) \\ &> 0 \end{aligned}$$

$$\begin{aligned} D_{n1} - D_{n2} &= 1/2[\{2lwy - (h_1 + y)(l + w - 1)\} - \{2lwy - (h_1 + 3 + y - 1)(l + w - 1)\}] \\ &= 1/2[\{2lwy - (h_1 + y)(l + w - 1)\} - \{2lwy - (h_1 + y + 2)(l + w - 1)\}] \\ &= 1/2[-(h_1 + y)(l + w - 1) + (h_1 + y + 2)(l + w - 1)] \\ &= (l + w - 1) \\ &> 0 \end{aligned}$$

The above results mean that when the value of h is 1 layer more than $[y(3.4142)]$, the number of spheres that can be added to P_n is the same as when h is exactly the multiple of $[y(3.4142)]$. The value of V_n is least when h is 3 layers more than $[y(3.4142)]$, or when h = $[y(3.4142)] + 3$. Therefore Vn is maximum and Pn is most dense when h is exactly the multiple of $[y(3.4142)]$, while Vn is minimum and Pn is least dense when h = $[y(3.4142)] + 3$, or when $h = [y(3.4142)] - 1$.

CONCLUSIONS AND RECOMMENDATIONS

From the proofs of the theorems, the conjecture is clearly proven. That is, it is possible for a parallelepiped box to contain more spheres than its maximum content when in default position provided that the dimensions of the box meet or exceed the required minimum of 5x3x4. Otherwise, the resulting pile, Pn, will contain lesser number of spheres than the content of the original pile, P_o. The new pile, P_n, attains its maximal content if its height is the multiple of $[y(3.4142)]$, and attains its minimal content if its height is equal to $[y(3.4142)] + 3$, or equal to $[y(3.4142)] - 1$.

Further study on the topic may be conducted in order to explore and exhaust other possibilities of obtaining new maximum that may exceed the foregoing results of the present study, as the present study is in no way exhaustive.

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